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Yang–Baxter matrices and differential calculi on quantum hyperplanes

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Abstract. It is shown that any invertible matrix R that solves the Yang–Baxter equations generates a set of quantum hyperplanes where differential calculi can be defined. The number of such quantum hyperplanes is given by the number of different eigenvalues of the matrix R. Several examples of two-dimensional quantum hyperplanes and differential calculi are presented. The relations of quantum hyperplanes and differential calculi are covariant WRT the quantum groups defined by the matrix R. In the generic cases the exterior differential satisfies the condition $d^2=0$.

1. Introduction

Recently, Wess and Zumino developed a differential calculus on quantum hyperplanes covariant WRT $\mathrm{GL}_q(N)$ [1]. They derived general constraints for matrices that determine commutation relations for variables, differentials and derivatives and found a solution of these constraints corresponding to the well-known quantum hyperplane given by the relations

$$x^i x^j = q x^j x^i \qquad i,j=1,\ldots,n \qquad i<j \qquad q\in\mathbb{C}. \tag{1.1}$$

In this paper we are going to show that any invertible matrix that solves the Yang–Baxter equation (YBE)

$$R_{12}R_{23}R_{12}=R_{23}R_{12}R_{23} \tag{1.2}$$

(braid group relations) can be used for definition of quantum hyperplanes and differential calculi on them. We are going to present examples of such quantum hyperplanes and differential calculi.

Wess and Zumino considered quantum hyperplanes generated by n variables $x^1,\ldots,x^n$ that satisfy quadratic relations (see also [2] and [3])

$$r^{ij}(x):=x^i x^j - B^{ij}{}_{kl}x^k x^l=0 \tag{1.3}$$

or, symbolically,

$$r_{12}(x):=x_1x_2-B_{12}x_1x_2=0 \tag{1.4}$$

where $B_{12}=\{B^{ij}{}_{kl}\}$ is a matrix with $n^2\times n^2$ elements in a field, e.g. in $\mathbb{C}$. They introduce differentials ξ^i ($\xi^i=\mathrm{d}x^i$) and derivatives ∂_i ($\partial_i x^j=\delta^j{}_i$) and require that the exterior differential

$$d:=\xi^i\partial_i \tag{1.5}$$

satisfies the Leibnitz rule

$$d(fg)=(df)g+f(dg) \tag{1.6}$$

for arbitrary functions f and g of x (i.e. for arbitrary elements of the quadratic algebra). Also, they assume commutation relations of variables and differentials in the form

$$x^i\xi^j = C^{ij}{}_{kl}\xi^k x^l \tag{1.7}$$

where C is again a matrix with $n^2 \times n^2$ numeric elements.

The Leibnitz rule then gives the commutation relations for ∂ and x,

$$\partial_j x^i = \delta^i{}_j + C^{ik}{}_{jl} x^l \partial_k \tag{1.8}$$

and two conditions for the matrices B and C,

$$(E_{12} - B_{12})(E_{12} + C_{12}) = 0 \tag{1.9}$$

$$(E_{12} - B_{12})C_{23}C_{12}x_2x_3 = 0 \tag{1.10}$$

where E is the $n^2 \times n^2$ unit matrix. The latter condition can be satisfied by the equation

$$B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23} \tag{1.11}$$

(even though weaker constraints can be required).

The algebra of x^i, ξ^j and ∂_k can be completed by relations

$$\partial_j\xi^i = (C^{-1})^{ik}{}_{jl}\xi^l\partial_k \qquad \xi^i\xi^j = -C^{ij}{}_{kl}\xi^k\xi^l \tag{1.12}$$

$$\partial_i\partial_j = F^{kl}{}_{ji}\partial_l\partial_k. \tag{1.13}$$

The consistency checks of (1.7), (1.8), (1.12) and (1.13) yield

$$C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23} \tag{1.14}$$

$$(E_{12} + C_{12})(E_{12} - F_{12}) = 0 \tag{1.15}$$

$$\partial_3\partial_2 C_{12}C_{23}(E_{12} - F_{12}) = 0. \tag{1.16}$$

Due to (1.13), the latter relation can be satisfied if

$$C_{12}C_{23}F_{12} = F_{23}C_{12}C_{23}. \tag{1.17}$$

Wess and Zumino solved constraints (1.9), (1.11), (1.14), (1.15) and (1.17) by B, C, F, which are multiples of a special matrix that satisfies the YBE. In the next section we are going to present more general solutions of the constraints.

Let us note that the condition $d^2 = 0$ was not used in the derivation of the constraints.

2. Solution of the constraints

Suppose that we have an invertible matrix $R = \{R^{ij}{}_{kl}\}$ with $n^2 \times n^2$ complex entries that solves the YBE (1.2). Let the minimal polynomial $M(x)$ of the matrix R be of degree m, i.e.

$$M(R) := (R - \lambda_1)(R - \lambda_2)\ldots(R - \lambda_m) = 0 \qquad (m \leqslant n^2). \tag{2.1}$$

(The roots of the polynomial M are equal to the eigenvalues of R but their multiplicities may differ.) Then it is easy to show that, for $k = 1, \ldots, m$, *the matrices*

$$C = C_k(R) = -R/\lambda_k \qquad B = B_k(R) = E - M_k(R)/K_k \qquad F = B \tag{2.2}$$

where

$$M_k(x) := (x - \lambda_k)^{-1}M(x) \qquad K_k = \prod_{j \neq k}(-\lambda_j) \tag{2.3}$$

(K_k *are convenient normalization coefficients*) *satisfy the constraints* (1.9), (1.11), (1.14), (1.15) and (1.17).

The constraint (1.14) is equivalent to the YBE for R. Equations (1.9) and (1.15) are satisfied due to (2.1). Constraints (1.11) and (1.17) are satisfied due to the fact that $B_k(R)$ and $F_k(R)$ are polynomials in R and any polynomial P solves the equation

$$P(R_{12})R_{23}R_{12}=R_{23}R_{12}P(R_{23}) \tag{2.4}$$

as a corollary of the YBE.

Moreover, *if* λ_k *is a simple root of the minimal polynomial then* $d^2=0$. Indeed, for any C and F we get, from (1.5), (1.12) and (1.7),

$$d^2=-\xi^i\xi^j\partial_j\partial_i=-F^{nl}{}_{ij}\xi^i\xi^j\partial_l\partial_n. \tag{2.5}$$

Due to (2.2) we then get

$$d^2=\left(1-K_k^{-1}\prod_{j\neq k}(\lambda_k-\lambda_j)\right)d^2 \tag{2.6}$$

so that if $\lambda_j\neq\lambda_k$ for $j\neq k$ then $d^2=0$.

Let us note that the matrices $E-B_k(R)$, which are used for the definition of the quantum hyperplanes, are nilpotent or proportional to projectors on the subspaces of eigenvectors corresponding to λ_k. Indeed, if $j\neq k$ then, due to (2.1), one gets

$$(E-B_k(R))(E-B_j(R))=0. \tag{2.7}$$

If $\lambda_j=\lambda_k$ for some $j\neq k$, i.e. if λ_k is a multiple root of M, then the matrix $E-B_k(R)$ is nilpotent. If λ_k is a simple root of M then $E-B_k(R)$ is proportional to the projector (onto the subspace of eigenvectors of R with eigenvalue λ_k) because

$$(E-B_k(R))^2=(E-B_k(R))\prod_{j\neq k}(1-\lambda_k/\lambda_j). \tag{2.8}$$

Moreover, if all roots of M are simple then

$$\sum_{k=1}^{m}(E-B_k(R))\prod_{j\neq k}(1-\lambda_k/\lambda_j)^{-1}=1. \tag{2.9}$$

By this method we obtain a good construction of quantum hyperplanes $Q_k(R)$ with differential calculi. The only ingredient we need is a matrix R satisfying the YBE. Actually, for a given R we have a whole class of quantum hyperplanes and differential calculi because the YBEs are invariant WRT the following transformations:

$$R\to\tilde{R}=(A\otimes A)R(A\otimes A)^{-1} \qquad A\in \mathrm{GL}(n) \tag{2.10}$$

$$R\to R^+=PRP \tag{2.11}$$

$$R\to R^-=R^{-1} \tag{2.12}$$

where $P^{ij}{}_{kl}=\delta^i{}_l\delta^j{}_k$. The quantum hyperplanes corresponding to $\tilde{R}$ are given by the quadratic relations obtained from those corresponding to R by the transformations $x^i\to A^i{}_jx^j$. The quantum hyperplanes corresponding to R^+ are obtained similarly by $x^ix^j\to x^jx^i$. The role of the last symmetry will be discussed later.

Let us investigate the special cases $m=1,2$.

Let $m=1$. Then $R=\lambda E$, $B=F=0$, $C=-E$ and the relations for variables, differentials and derivatives are

$$x^i x^j=0 \qquad x^i\xi^j=-\xi^j x^i \tag{2.13}$$

$$\partial_j x^i=\delta^i{}_j(1-x^k\partial_k) \qquad \partial_j\xi^i=-\delta^i{}_j\xi^l\partial_l \qquad \partial_i\partial_j=0. \tag{2.14}$$

Note that in this trivial case there are no quadratic relations for differentials ξ^i.

Let $m=2$. Then the matrix R satisfies the Hecke condition

$$R^2=\alpha R+\beta \tag{2.15}$$

and the quantum hyperplanes and the differential calculi are given by

$$C_1=-R/\lambda_1 \qquad B_1=F_1=R/\lambda_2 \tag{2.16}$$

$$C_2=-R/\lambda_2 \qquad B_2=F_2=R/\lambda_1. \tag{2.17}$$

Note that the commutation relations for variables in one quantum hyperplane (given by B_i) are the same as commutation relations for differentials (given by C_i) in the other.

Another remarkable fact is that if $\lambda_1+\lambda_2\neq 0$ then there are (at least) two different differential calculi for one quantum hyperplane. Indeed, if R is a solution of the YBE then, as mentioned above, R^{-1} is also a solution and the corresponding quantum hyperplanes and differential calculi are defined by

$$B_i(R^{-1})=B_i^{-1}(R) \qquad C_i(R^{-1})=C_i^{-1}(R) \qquad F_i(R^{-1})=F_i^{-1}(R). \tag{2.18}$$

The quantum hyperplane defined by a matrix B_i^{-1} is identical with that defined by B_i. However, the differential calculi are different except for the case when $C_i=C_i^{-1}$, i.e. $R^2=\lambda_i^2$.

If the minimal polynomial of a solution of the YBE is of degree higher than two then the above given construction of the differential calculi is not applicable to all quantum hyperplanes that can be obtained from the solution. The reason is that any product $(R-\lambda_{i_1})\dots(R-\lambda_{i_n})$, $n<m$, $i_j\neq i_k$ where λ_{i_j} are roots of the minimal polynomial, defines a quantum hyperplane [5]; however, the construction works only for the hyperplanes given by $M_k(R)$. Definition of the differential calculi on the other hyperplanes is an open problem.

In the next section we shall give examples of quantum hyperplanes where the described construction of differential calculi can be applied.

3. Two-dimensional examples

In a previous paper [4] we gave a list of eight (and less) vertex solutions of the YBE. In [5], the two-dimensional quantum hyperplanes obtainable from these solutions, i.e. associative quadratic algebras [2, 3]

$$Q_f(R):=\mathbb{C}\langle x^1,x^2\rangle/f(R)(x\otimes x) \tag{3.1}$$

where f is a singular polynomial of R, were presented. We can use the above given prescription for B, C, F to define the differential calculi on many of these spaces. Below we shall show several illustrative examples. The number of the R-matrices corresponds to [5] and $x^1\equiv x$, $x^2\equiv y$, $\xi^1\equiv\xi$, $\xi^2\equiv\eta$.

(i) The two-parametric generalization of the example in [1]. Let

$$R = R_5 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-t & qt & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad qt \neq 0. \tag{3.2}$$

(See also [6–8].)

The minimal polynomial for R_5 is of second order,

$$(R_5 - q)(R_5 + t) = 0. \tag{3.3}$$

so that the quantum hyperplanes $Q_i(R_5)$, corresponding to $B_i(R_5)$ $i = 1, 2$ and the corresponding differential calculi are given by matrices (2.15) and (2.16). The relations (1.3), (1.7), (1.8), (1.12) and (1.13) then read

$$(q-\tilde{\lambda}_i)x^2 = 0 \qquad (q-\tilde{\lambda}_i)y^2 = 0 \qquad xy = \tilde{\lambda}_i yx \tag{3.4}$$

$$(q-\lambda_i)\xi^2 = 0 \qquad (q-\lambda_i)\eta^2 = 0 \qquad \xi\eta = \lambda_i \eta\xi \tag{3.5}$$

$$(q-\tilde{\lambda}_i)\partial_x^2 = 0 \qquad (q-\tilde{\lambda}_i)\partial_y^2 = 0 \qquad \partial_x\partial_y = -\lambda_i\partial_y\partial_x \tag{3.6}$$

$$\begin{aligned} &\lambda_i x\xi = -q\xi x \qquad \lambda_i y\eta = -q\eta y \\ &\lambda_i x\eta = \xi y(t-q) - qt\eta x \qquad \lambda_i y\xi = -\xi y \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\partial_x x = 1 - \lambda_i^{-1}[qx\partial_x + (q-t)y\partial_y] \qquad \partial_x y = -\lambda_i^{-1} y\partial_x \\ &\partial_y x = -\lambda_i^{-1} qtx\partial_y \qquad \partial_y y = 1 - \lambda_i^{-1} qy\partial_y \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\partial_x\xi = -\lambda_i q^{-1}\xi\partial_x \qquad \partial_x\eta = -\lambda_i q^{-1}t^{-1}\eta\partial_x \\ &\partial_y\xi = -\lambda_i\xi\partial_y \qquad \partial_y\eta = \lambda_i[(t^{-1}-q^{-1})\xi\partial_x - q^{-1}\eta\partial_y] \end{aligned} \tag{3.9}$$

where $\lambda_1 = \tilde{\lambda}_2 = -t$, $\lambda_2 = \tilde{\lambda}_1 = q$.

One can see from (3.4) that the quantum hyperplane $Q_1(R_5)$ is given by the well-known relation $xy = qyx$. On the other hand, $Q_2(R_5)$ in the generic case $q \neq -t$, is a four-dimensional algebra with the basis 1, x, y, $xy = -tyx$.

Note that even though the quantum hyperplanes are given by only one of the parameters q, t the formulae for the differential calculi also contain the other one. It means that, in the generic cases $q \neq -t$, one-parametric sets of differential calculi can be defined on both the quantum hyperplanes $Q_i(R_5)$. The differential calculus presented in [1] is obtained for $t = 1/q$.

If $q = -t$ then the matrix R_5 determines just one differential calculus on $Q_1(R_5) = Q_2(R_5)$.

The bases in the quantum hyperplanes are formed by the monomials $y^n x^m$ where $n, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ for λ_1 and $n, m \in \{0, 1\}$ for $\lambda_2 \neq \lambda_1$. If $q = -t$ then $n, m \in \mathbb{N}_0$. The derivatives of the basis elements are

$$\begin{aligned} &\partial_x y^n x^m = y^n x^{m-1}(-\lambda_1)^{-n} G_m(-q/\lambda_i) \\ &\partial_y y^n x^m = y^{n-1} x^m G_n(-q/\lambda_i) \end{aligned} \tag{3.10}$$

where

$$G_m(x) := (1 - x^m)/(1 - x). \tag{3.11}$$

As $R(q,t)^{-1} = PR(q^{-1}, t^{-1})P$, the formulae for the alternative differential calculi corresponding to R^{-1} can be obtained by $q \to 1/q$, $t \to 1/t$ and $wz \to zw$, where $w, z \in \{x, y, \xi, \eta, \partial_x, \partial_y\}$.

(ii) Grassmanian quantum hyperplanes (see also [9-11]). Let

$$R = R_6 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-t & qt & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \qquad qt \neq 0 \qquad q \neq -t. \tag{3.12}$$

The minimal polynomial is again of second order and

$$\lambda_1 = q \qquad \lambda_2 = -t. \tag{3.13}$$

The relations defining the quantum hyperplane and the differential calculus corresponding to $B_1(R_6)$ are given by

$$x^2 = 0 \qquad xy = -tyx \tag{3.14}$$

$$\eta^2 = 0 \qquad \xi\eta = q\eta\xi \tag{3.15}$$

$$\partial_x\partial_x = 0 \qquad \partial_x\partial_y = -q\partial_y\partial_x \tag{3.16}$$

while for $B_2(R_6)$ we get

$$y^2 = 0 \qquad xy = qyx \tag{3.17}$$

$$\xi^2 = 0 \qquad \xi\eta = -t\eta\xi \tag{3.18}$$

$$\partial_y\partial_y = 0 \qquad \partial_x\partial_y = t\partial_y\partial_x. \tag{3.19}$$

The remaining relations for both the hyperplanes are

$$\begin{aligned} &\lambda_i x\xi = -q\xi x \qquad \lambda_i y\eta = t\eta y \\ &\lambda_i x\eta = \xi y(t-q) - qt\eta x \qquad \lambda_i y\xi = -\xi y \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\partial_x x = 1 + \lambda_i^{-1}[(t-q)y\partial_y - qx\partial_x] \qquad \partial_x y = -\lambda_i^{-1} y\partial_x \\ &\partial_y x = -\lambda_i^{-1} qtx\partial_y \qquad \partial_y y = 1 + \lambda_i^{-1} ty\partial_y \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\partial_x\xi = -\lambda_i q^{-1}\xi\partial_x \qquad \partial_x\eta = -\lambda_i q^{-1}t^{-1}\eta\partial_x \\ &\partial_y\xi = -\lambda_i\xi\partial_y \qquad \partial_y\eta = \lambda_i[(t^{-1} - q^{-1})\xi\partial_x + t^{-1}\eta\partial_y]. \end{aligned} \tag{3.22}$$

Monomials that form bases in the algebras $Q_i(R_6)$, $i = 1, 2$ given by (3.14) respectively (3.17) are $y^n x^m$ where $n \in \mathbb{N}_0$, $m \in \{0, 1\}$ for λ_1 and $n \in \{0, 1\}$, $m \in \mathbb{N}_0$ for λ_2. The derivative rules for the basis monomials are

$$\begin{aligned} &\partial_x y^n x^m = y^n x^{m-1}(-\lambda_i)^{-n} G_m(-q/\lambda_i) \\ &\partial_y y^n x^m = y^{n-1} x^m G_n(t/\lambda_i). \end{aligned} \tag{3.23}$$

Again, $R(q,t)^{-1} = PR(q^{-1}, t^{-1})P$ so that the formulae for the alternative differential calculi can be obtained by $q \to 1/q$, $t \to 1/t$, and the reverse of factors in products.

(iii) The light-cone quantum hyperplanes. Let

$$R = R_2 = \begin{pmatrix} 1+t & 0 & 0 & r \\ 0 & 1 & s & 0 \\ 0 & s & 1 & 0 \\ r^{-1} & 0 & 0 & 1-t \end{pmatrix} \qquad s^2 = 1 + t^2 \qquad r, t \neq 0. \tag{3.24}$$

(See also [12].)

The minimal polynomial for R_2 is of order $m=2$ and

$$\lambda_1=\lambda_+=1+s \qquad \lambda_2=\lambda_-=1-s. \tag{3.25}$$

The quantum hyperplanes $Q_\pm(R_2)$ corresponding to $B_\pm=R_2/\lambda_\mp$ are given by

$$(t\pm s)x^2+ry^2=0 \qquad s(xy\pm yx)=0 \tag{3.26}$$

and the differential calculi on $Q_\pm(R_2)$ are defined by the relations

$$\xi^2(t\mp s)+r\eta^2=0 \qquad s(\xi\eta\mp\eta\xi)=0 \tag{3.27}$$

$$r(t\pm s)\partial_x\partial_x+\partial_y\partial_y=0 \qquad s(\partial_x\partial_y\pm\partial_y\partial_x)=0 \tag{3.28}$$

$$\begin{aligned} &-\lambda_\pm x\xi=\xi x(1+t)+r\eta y && -\lambda_\pm y\eta=\eta y(1-t)+r^{-1}\xi x\\ &-\lambda_\pm x\eta=\xi y+s\eta x && -\lambda_\pm y\xi=s\xi y+\eta x \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\partial_x x=1-\lambda_\pm^{-1}[(1+t)x\partial_x+y\partial_y] && \partial_x y=-\lambda_\pm^{-1}(sy\partial_x+r^{-1}x\partial_y)\\ &\partial_y x=-\lambda_\pm^{-1}(sx\partial_y+ry\partial_x) && \partial_y y=1-\lambda_\pm^{-1}[x\partial_x+(1-t)y\partial_y] \end{aligned} \tag{3.30}$$

$$\begin{aligned} &-\lambda_\mp\partial_x\xi=(1-t)\xi\partial_x+\eta\partial_y && \lambda_\mp\partial_x\eta=s\eta\partial_x+r^{-1}\xi\partial_y\\ &\lambda_\mp\partial_y\xi=s\xi\partial_y+r\eta\partial_x && -\lambda_\mp\partial_y\eta=\xi\partial_x+\eta\partial_y(1+t). \end{aligned} \tag{3.31}$$

Note that $R^{-1}=-t^{-2}R(-t,-s,-r)$. Therefore, the alternative differential calculi given by B^{-1}, C^{-1}, F^{-1} are the same as above up to $t\to -t$, $s\to -s$, $r\to -r$.

Due to (3.26), functions on the hyperplanes, or more precisely elements of the algebras $Q_\pm(R_2)$, are linear combinations of x^n and $x^n y$ where $n\in\mathbb{N}_0\equiv\{0,1,2,\ldots\}$. Derivatives of these monomials are rather complicated and I was not able to derive the general formulae.

(iv) Another interesting example that in a way is a mixture of the two preceding ones is given by the matrix

$$R=R_3=\begin{pmatrix}1&0&0&0\\0&1-t&st&0\\0&s&0&0\\1&0&0&-t\end{pmatrix} \qquad s^2=1 \qquad t\neq 0. \tag{3.32}$$

The minimal polynomial condition for R_3 is

$$(R_3+t)(R_3-1)=0. \tag{3.33}$$

The quantum hyperplanes and the differential calculi are given by

$$x^2=(1-\lambda_i)y^2 \qquad \lambda_i xy=-styx \tag{3.34}$$

$$\xi^2=(1-\tilde\lambda_1)\eta^2 \qquad \tilde\lambda_i\xi\eta=-st\eta\xi \tag{3.35}$$

$$\tilde\lambda_i\partial_x\partial_y=st\partial_y\partial_x \qquad (\tilde\lambda_i-1)\partial_x\partial_x=\partial_y\partial_y \tag{3.36}$$

$$\begin{aligned} &\lambda_i x\xi=-\xi x && \lambda_i y\eta=t\eta y-\xi x\\ &\lambda_i x\eta=\xi y(t-1)-st\eta x && \lambda_i y\xi=-s\xi y \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\partial_x x=1+\lambda_i^{-1}[ty\partial_y-(x\partial_x+y\partial_y)] && \partial_x y=-\lambda_i^{-1}(sy\partial_x+x\partial_y)\\ &\partial_y x=-\lambda_i^{-1}stx\partial_y && \partial_y y=1+\lambda_i^{-1}ty\partial_y \end{aligned} \tag{3.38}$$

$$\begin{aligned} &\partial_x\xi=-\lambda_i\xi\partial_x && \partial_x\eta=-\lambda_i t^{-1}(s\eta\partial_x+\xi\partial_y)\\ &\partial_y\xi=-\lambda_i s\xi\partial_y && \partial_y\eta=\lambda_i t^{-1}[(1-t)\xi\partial_x+\eta\partial_y] \end{aligned} \tag{3.39}$$

where $\lambda_1=\tilde\lambda_2=1$, $\lambda_2=\tilde\lambda_1=-t$.

One can see from (3.34) that $Q_1(R_3)$ is the quantum hyperplane with a Grassmanian variable and the hyperplane $Q_2(R_3)$ is of the light-cone type (for $t\neq -1$). However, the differential calculi are substantially different from those in examples (ii) and (iii).

Monomials that form a convenient basis in the algebra given by (3.35) are y^n and $y^n x$. The derivative rules for the basis monomials are in this case

$$\begin{aligned}
&\partial_x y^n = -y^{n-2}x(-s/\lambda_i)^n S_n(t,\lambda_i)\\
&\partial_x y^n x = y^n(-s/\lambda_i)^n[1-(1-\lambda_i)S_n(t,\lambda_i)]\\
&\partial_y y^n = y^{n-1}G_n(t/\lambda_i) \qquad \partial_y y^n x = y^{n-1}xG_n(t/\lambda_i)
\end{aligned} \tag{3.40}$$

where

$$S_n(t,\lambda_i) := \lambda_i^2(\lambda_i t - t^2)^{-1}[G_n(t) - G_n(t^2/\lambda_i)]. \tag{3.41}$$

In particular,

$$S_n(t,\lambda_1=1) = (1-t^n)(1-t^{n-1})(1-t)^{-2}(1+t)^{-1}$$

$$S_n(t,\lambda_2=-t) = (1-t^2)^{-1}\{t^n[1-(-1)^n] - t^{n+1}[1-(-1)^{n+1}] - 2t\}/2.$$

(v) Let

$$R = R_9 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & t & 0 & 0\\ 0 & 0 & t & 0\\ 1 & 0 & 0 & 0\end{pmatrix} \qquad t^2 \neq 1, 0. \tag{3.42}$$

The minimal polynomial for R_9 is of the third degree and

$$\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad \lambda_3 = t. \tag{3.43}$$

The quantum hyperplanes corresponding to these eigenvalues are

$$Q_{\pm 1}(R_9) := \mathbb{C}\langle x, y\rangle/(x^2 \pm y^2 = 0) \tag{3.44}$$

$$Q_t(R_9) := \mathbb{C}\langle x, y\rangle/(xy = yx = 0). \tag{3.45}$$

There are also other quantum hyperplanes defined by this matrix R but one cannot define the differential calculi on them by the above given construction because the matrices B that correspond to these spaces are not of the form (2.2). The differential calculi for the spaces (3.44) and (3.45) are given by

$$\begin{aligned}
&x\xi = -\lambda_i^{-1}\eta y \qquad y\eta = -\lambda_i^{-1}\xi x\\
&x\eta = -\lambda_i^{-1}t\xi y \qquad y\xi = -\lambda_i^{-1}t\eta x
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
&\partial_x x = 1 - \lambda_i^{-1}ty\partial_y \qquad \partial_x y = -\lambda_i^{-1}x\partial_y\\
&\partial_y x = -\lambda_i^{-1}y\partial_x \qquad \partial_y y = 1 - \lambda_i^{-1}tx\partial_x
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
&\partial_x \xi = -\lambda_i t^{-1}\eta\partial_y \qquad \partial_x\eta = -\lambda_i\xi\partial_y\\
&\partial_y\xi = -\lambda_i\eta\partial_x \qquad \partial_y\eta = -\lambda_i t^{-1}\xi\partial_x
\end{aligned} \tag{3.48}$$

and

$$\xi^2 = \pm\eta^2 \qquad \xi\eta = \eta\xi = 0 \tag{3.49}$$

$$\partial_x\partial_x \pm \partial_y\partial_y = 0. \tag{3.50}$$

for the hyperplanes $Q_{\pm 1}(R_9)$ or

$$\xi^2 = \eta^2 = 0 \tag{3.51}$$

$$\partial_x\partial_y = \partial_y\partial_x = 0 \tag{3.52}$$

for the space $Q_t(R_9)$.

The derivative rules for monomials x^n and y^n forming the basis in $Q_t(R_9)$ are

$$\begin{aligned} &\partial_x x^{2m} = x^{2m-1} \qquad \partial_x x^{2m+1} = x^{2m} - y^{2m} t^{-m} \\ &\partial_y x^{2m} = -y^{2m-1} t^{-m} \qquad \partial_y x^{2m-1} = 0 \qquad m = 1, 2, \ldots . \end{aligned} \tag{3.53}$$

We have not derived the derivative rules for the monomials $x^n(yx)^m y^r$, n, $m \in \mathbb{N}_0$, $r \in \{1, 0\}$ that form the bases in $Q_{\pm 1}(R_9)$.

(vi) The minimal polynomial with multiple root. Let

$$R = R_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & s & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad s^2 = 1. \tag{3.54}$$

The minimal polynomial of R_7 is

$$(R_7 - 1)^2 (R_7 + 1) = 0. \tag{3.55}$$

The root $\lambda_1 = 1$ is multiple and therefore the matrix $E - B_1(R)$ is nilpotent. The quantum hyperplanes corresponding to $\lambda_i = \pm 1$ are

$$Q_{+1}(R_7) := \mathbb{C}\langle x, y \rangle / (x^2 = 0) \tag{3.56}$$

$$Q_{-1}(R_7) := \mathbb{C}\langle x, y \rangle / (xy = syx). \tag{3.57}$$

The differential calculi on $Q_{\pm 1}(R_7)$ are given by

$$\begin{aligned} &x\xi = \mp \xi x \qquad y\eta = \mp(\xi x + \eta y) \\ &x\eta = \mp s\eta x \qquad y\xi = \mp s\xi y \end{aligned} \tag{3.58}$$

$$\begin{aligned} &\partial_x x = 1 \mp x\partial_x \qquad \partial_x y = \mp(x\partial_y + sy\partial_x) \\ &\partial_y x = \mp sx\partial_y \qquad \partial_y y = 1 \mp y\partial_y \end{aligned} \tag{3.59}$$

$$\begin{aligned} &\partial_x \xi = \mp \xi \partial_x \qquad \partial_x \eta = \mp s\eta\partial_x \pm \xi\partial_y \\ &\partial_y \xi = \mp s\xi\partial_y \qquad \partial_y \eta = \mp \eta\partial_y \end{aligned} \tag{3.60}$$

and, besides these relations, it holds that

$$\xi^2 = 0 \qquad \xi\eta - s\eta\xi = 0 \qquad \partial_y\partial_y = 0 \tag{3.61}$$

on $Q_{+1}(R_7)$ and

$$\xi^2 = \eta^2 = 0 \qquad \xi\eta + s\eta\xi = 0 \qquad \partial_x\partial_y - s\partial_y\partial_x = 0 \tag{3.62}$$

on $Q_{-1}(R_7)$.

We have not derived the derivative rules for the monomials $y^{n_0}xy^{n_1}x \ldots y^{n_{K-1}}xy^{n_K}$, K, n_0, $n_K \in \mathbb{N}_0$, $n_1, \ldots n_{K-1} \in \mathbb{N}$ that form the basis in $Q_{+1}(R_7)$.

The derivative rules for monomials $x^n y^m$ forming the basis in $Q_{-1}(R_7)$ are

$$\partial_x x^n y^m = nx^{n-1}y^m + \binom{m}{2} x^{n+1} y^{m-2} \qquad \partial_y x^n y^m = s^n m x^n y^{m-1}. \tag{3.63}$$

Note that even though $Q_{-1}(R_7)$ is a hyperplane with (anti) commuting variables the differential calculus given by R_7 is rather peculiar.

4. Conclusions

We have presented a construction of quantum hyperplanes and differential calculi based on R-matrices that solve the YBE. To any eigenvalue of the matrix R correspond a quantum hyperplane and a differential calculus. The formulae for matrices B, C, F that define their structure are given by (2.1)–(2.3).

One can see from the examples in section 3 that there may exist several differential calculi for one quantum hyperplane because the hyperplane can be defined by various matrices R that give different differential calculi.

A surprising fact is that even though the condition for exterior differential $d^2=0$ was not required, it is satisfied except for the quantum hyperplanes and differential calculi (given by B_i, C_i, F_i) that correspond to multiple roots λ_i of the minimal polynomial.

The last remark concerns the covariance of the differential calculi. It is easy to see that, as the matrices B, C are expressed in terms of R, the commutation relations concerning only variables or differentials are invariant WRT the transformations

$$x^{i'} = T^i{}_j x^j \qquad \xi^{i'} = T^i{}_j \xi^j \tag{4.1}$$

where $T^i{}_j$ are non-commuting elements satisfying relations of the quantum group

$$R^{ij}{}_{kl} T^k{}_m T^l{}_n = T^i{}_k T^j{}_l R^{kl}{}_{mn}. \tag{4.2}$$

On the other hand, the covariance of the relations for x, ξ often implies the relations of the quantum group [5]. A little more complicated is the covariance of the relations containing derivatives because they transform by the inverse matrix

$$\partial_i' = \partial_j \tilde{T}^j{}_i \qquad T\tilde{T} = \tilde{T}T = E. \tag{4.3}$$

The existence of such an inverse matrix (with non-commuting elements) is not obvious. Nevertheless, it is claimed in [3] (remark 16) that any algebra given by (4.3) can be extended in such way that the antipod may be defined, i.e. the inverse matrix does exist in the extended algebra.

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References

[1] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Preprint* CERN-TH-5697-90, LAPP-TH-284-90
[2] Manin I Yu 1988 *Quantum Groups and Non-commutative Geometry* (Montreal: CRM, Université Montréal)
[3] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Algebra Analiz* **1** 178 (in Russian)
[4] Hlavatý L 1987 *J. Phys. A: Math. Gen.* **20** 1661
[5] Hlavatý L 1991 *J. Phys. A: Math. Gen.* **24** 2903
[6] Schrirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317
[7] Sudbery A 1990 *J. Phys. A: Math. Gen.* **23** L697
[8] Burdík Č and Hlavatý L 1991 *J. Phys. A: Math. Gen.* **24** L165
[9] Schwenk J, Schmidke W B and Vokos S P 1990 *Z. Phys.* C **46** 643
[10] Soni S K 1991 *J. Phys. A: Math. Gen.* **24** 619
[11] Jing N, Ge M L and Wu Y S 1991 *Lett. Math. Phys.* **21** 193
[12] Ge M-L, Gwa L-H and Zhao H-K 1991 *J. Phys. A: Math. Gen.* **23** L795